# CAT 4

(Functoriality)

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, dashed] & \bullet \arrow[d, dashed] \\ \circ \arrow[r] & \circ \end{tikzcd}$$

**Goal 0** To understand how simplicial maps $f : K \to L$ induce linear maps of homology groups $H_i f : H_i K \to H_i L$.

**Def 1** Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be chain complexes. A CHAIN MAP is a sequence $\varphi_i\ C_i \to C_i'$ of linear maps which makes all the squares commute:

$$\begin{tikzcd} \cdots \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] \arrow[d, dashed, "\varphi_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, dashed, "\varphi_1"] & C_0 \arrow[d, dashed, "\varphi_0"] \\ \cdots \arrow[r, "\partial_3'"'] & C_2' \arrow[r, "\partial_2'"'] & C_1' \arrow[r, "\partial_1'"'] & C_0' \end{tikzcd}$$

i.e, we have $\left(\varphi_i \circ \partial_{i+1} = \partial'_{i+1} \circ \varphi_{i+1}\right)\ \forall i \geq 0$

**Prop 2** Every chain map $\varphi_\bullet : (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ induces well-defined maps of homology groups $H_i\varphi : H_i C \to H_i C'$ $\forall i \geq 0$

**Pf**

- We defined $H_i C$ as $Z_i/B_i$, where $= \ker \partial_i$ and $B_i = \operatorname{im} \partial_{i+1}$. Now if $\gamma \in Z_i$, then $\partial_i \gamma = 0$, which forces

$$0 = \varphi_{i-1}\partial_i(\gamma) = \partial_i' \varphi_i(\gamma),$$

so $\varphi_i(\gamma) \in \ker \partial_i' = Z_i'$. So, $\boxed{\varphi \text{ SENDS CYCLES TO CYCLES}}$

- Next, if $\beta \in B_i$ then $\exists \bar{\beta} \in C_{i+1}$ with $\partial_{i+1}\bar{\beta} = \beta$. So,

$$\varphi_i \partial_{i+1}(\bar{\beta}) = \partial'_{i+1} \circ \varphi_{i+1}(\bar{\beta}),$$

so $\varphi_{i+1}(\beta) \in \operatorname{im} \partial_{i+1} = B_i$. So, $\boxed{\varphi \text{ SENDS BOUNDARIES TO BOUNDARIES}}$. Thus, the assignment

This is the equiv class of a cycle $\gamma$

$$[\gamma] \longmapsto [\varphi(\gamma)] \quad \text{for cycles } \gamma \text{ (up to boundary)}$$

induces a well-defined map $H_i C \to H_i C'$.

**Q** What does this have to do with simplicial maps?

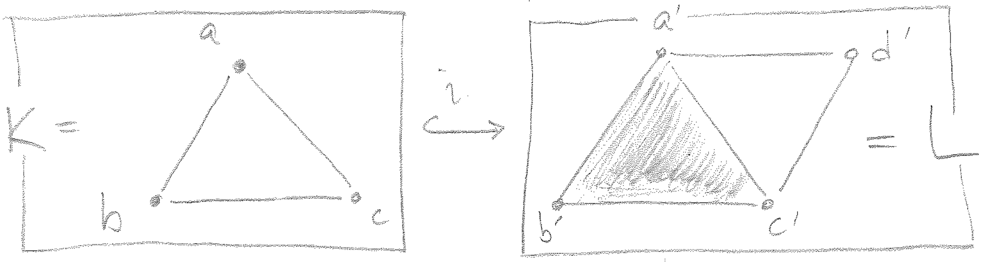
**Def 3** Let $f: K \to L$ be a simplicial map. For each dimension $i \geq 0$, define the chain map $C_i f: C_i^K \to C_i^L$ by

$$\boxed{C_i f(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = \dim \sigma \\ 0 & \text{otherwise.} \end{cases}}$$

CHECK that this satisfies Def 1 !!

[Here $C_i^K$ = vector space generated by $i$-simplices of $K$, and similarly for $C_i^L$. This map sends basis elements to basis elements when possible, otherwise to zero.

**Eg 4** Consider the "inclusion" map




$i(a) = a'$ etc.

Although both $K$ and $L$ have isomorphic homology, the map $H_1 i$ is <u>NOT</u> an isomorphism

**Thm 5** "functoriality" Let $K \xrightarrow{f} L \xrightarrow{g} M$ be two simplicial maps. Then $H_i(g \circ f)$ is the same as $H_i g \circ H_i f$, i.e. homology respects composition. [This will be <u>CRUCIAL</u> later...]

**Pf** This works in two stages: first show that the desired result holds for induced chain maps, i.e. that $C_i(g \circ f) = C_i g \circ C_i f$. [You first have to define composition of chain maps to make sense of the right side!]. Once you've done this, it only remains to check that for any $[\gamma] \in H_i K$, we get

$$H_i(g \circ f)[\gamma] = [C_i(g \circ f)(\gamma)]$$
$$= [C_i g \circ C_i f(\gamma)] = H_i g \circ H_i f([\gamma]).$$

**Q** If you have two chain maps $\varphi_\bullet, \psi_\bullet : C_\bullet \to C'_\bullet$, when do they induce identical maps on homology?

**Def 6** We say $\varphi_\bullet, \psi_\bullet : C_\bullet \to C'_\bullet$ are $\boxed{\text{CHAIN HOMOTOPIC}}$ if there are linear maps

$$\boxed{\Theta_i : C_i \longrightarrow C'_{i+1}} \quad \forall\, i \geq 0$$

satisfying the equality

$$\boxed{\varphi_i - \psi_i = \Theta_{i-1} \circ \partial_i + \partial'_{i+1} \circ \Theta_i}$$

$$\begin{tikzcd}
C_2 \arrow[r, "\partial_2"] \arrow[d, "\psi_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\psi_1"] \arrow[dl, "\Theta_1"'] & C_0 \arrow[d, shift right, "\varphi_0"'] \arrow[d, shift left, "\psi_0"] \arrow[dl, "\Theta_0"'] \\
C'_2 \arrow[r, "\partial'_2"'] & C'_1 \arrow[r, "\partial'_1"'] & C'_0
\end{tikzcd}$$

[Note that the $\Theta_i$ do NOT have to commute with any existing maps]

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**Prop 7** a) Chain homotopic maps induce the same maps on homology, i.e., if $\varphi_\bullet$ is chain homotopic to $\psi_\bullet$ then $H_\bullet\varphi = H_\bullet\psi$

b) If $\tilde{f}, \tilde{g} : |K| \to |L|$ are homotopic (in the usual sense of continuous functions), then $\exists$ simplicial complexes $K'$ and $L'$ plus simplicial maps $f, g : K' \to L'$ so that $|K'| = |K|$, $|L'| = |L|$, and $C_\bullet f$ is chain-homotopic to $C_\bullet g$.

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**Pf** a) is easy, b) is hard. [so we ignore it]. If $\gamma$ is a cycle in $C_i$, then

$$\varphi_i(\gamma) - \psi_i(\gamma) = \underbrace{\Theta_{i-1} \circ \partial_i(\gamma)}_{\substack{=0, \text{ since } \gamma \text{ is a} \\ \text{cycle in } C_\bullet}} + \underbrace{\partial'_{i+1} \circ \Theta_i(\gamma)}_{\text{a boundary in } C'_\bullet}$$

So, the difference is always a boundary, which is undetectable by homology.

# SEQUENCES

**Note** When studying homotopy, we got a LOT of important tools by asking "which spaces have trivial homotopy"? Now we will do the same by asking "which complexes have zero homology?"

**Def 8** We say that a sequence $(\cdots \to V_i \xrightarrow{\alpha_i} V_{i+1} \xrightarrow{\alpha_{i-1}} \cdots)$ of vector spaces is <u>EXACT</u> if $\ker \alpha_i = \operatorname{im} \alpha_{i+1}$ $\forall i$. [This implies we have a chain complex with all $H_i = 0$]

**Eg 9** Exactness encodes <u>many</u> standard notions:

Work even for (say) abelian groups:

- (i) $0 \to C_1 \xrightarrow{d_1} C_0$ exact $\Leftrightarrow$ $d_1$ injective.
- (ii) $C_1 \xrightarrow{d_1} C_0 \to 0$ exact $\Leftrightarrow$ $d_1$ surjective
- (iii) $0 \to C_1 \xrightarrow{d_1} C_0 \to 0$ exact $\Leftrightarrow$ $d_1$ isomorphism

for vector spaces only:

- (iv) $0 \to C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_0 \to 0$ exact $\Leftrightarrow$ $C_1 \simeq C_2 \oplus C_0$

**Def 9** A SHORT EXACT SEQUENCE of chain <u>complexes</u> is

$$\boxed{0 \longrightarrow C_\bullet \xrightarrow{\alpha_\bullet} D_\bullet \xrightarrow{\beta_\bullet} E_\bullet \longrightarrow 0}$$

where $C_\bullet, D_\bullet, E_\bullet$ are chain complexes and $\alpha_\bullet, \beta_\bullet$ are chain maps, so that $\alpha_\bullet$ is injective and $\beta_\bullet$ is surjective. [So, eg $\alpha_i : C_i \to D_i$ is injective $\forall i$, etc]

**Thm 10 (Big Deal)** Given a SES of chain complexes,

$$0 \longrightarrow C_\bullet \xrightarrow{\alpha_\bullet} D_\bullet \xrightarrow{\beta_\bullet} E_\bullet \longrightarrow 0,$$

there exists a family of linear maps

$$\boxed{\Delta_i : H_i E \longrightarrow H_{i-1} C}$$

called CONNECTING HOMOMORPHISMS, which fit into a <u>LONG exact sequence</u> of vector spaces like this:

$$\cdots \xrightarrow{\Delta_{i+1}} H_i C \xrightarrow{H_i \alpha} H_i D \xrightarrow{H_i \beta} H_i E \xrightarrow{\Delta_i} H_{i-1} C \xrightarrow{H_{i-1}\alpha} \cdots$$

(keeps going to $i=0$)

The proof of this is tedious and not too enlightening, BUT the applications are AMAZING. I'll discuss the two most important ones here (but there are LOTS...)

**Def 11.** Let $L$ be a subcomplex of a simplicial complex $K$. Writing $C_i^{K,L}$ for the <u>quotient</u> space $C_i^K / C_i^L$, we get a SES of chain complexes:

$$0 \to C_\bullet^L \xrightarrow{\alpha_\bullet} C_\bullet^K \xrightarrow{\beta_\bullet} C_\bullet^{K,L} \to 0,$$

where $\alpha_\bullet$ is the inclusion of chains and $\beta_\bullet$ is the projection map that sends $C_\bullet^L$ to zero. Now there is a long exact sequence

(This is the homology of K RELATIVE to L.)

$$\cdots \xrightarrow{\Delta_{i-1}} H_i(L) \xrightarrow{H_i\alpha} H_i(K) \xrightarrow{H_i\beta} \boxed{H_i(K,L)} \xrightarrow{\Delta_0} H_{i-1}(L) \to \cdots$$

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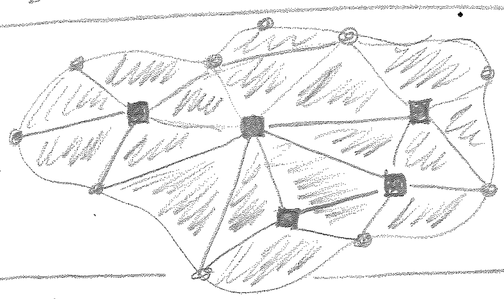
Eg 12 Let $K$ = triangulated disk and $L$ = $n$ internal vertices, eg

$K=$  $L$ = $n$ vertices labelled ■ ($n=5$ in this picture)

<u>What is $H_i(K,L)$?</u> Well, we know

$$H_i(K) = \begin{cases} \mathbb{F}, & i=0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad H_i(L) = \begin{cases} \mathbb{F}^n, & i=0 \\ 0, & \text{otherwise} \end{cases}$$

Here comes the LES:

$$0 \to H_1(K,L) \to H_0(L) \to H_0(K) \to H_0(K,L) \to 0$$

$\varnothing$ $\quad \#=\underline{n-1} \quad \#=n \quad \#=1 \quad \varnothing$

All we need here is the realization that the inclusion map $L \subseteq K$ sends $\mathbb{F}^n \to \mathbb{F}$ via a rank one map, eg $[\pm 1 \ \pm 1, \ldots, \pm 1]$. By exactness, this forces

$$\boxed{H_i(K,L) = \begin{cases} \mathbb{F}^{n-1} & i=1 \\ 0 & \text{otherwise} \end{cases}}$$

**Def 13**

a) Let $K$ be a simplicial complex given by the union $K = A \cup B$ of subcomplexes with intersection $I = A \cap B$. Then the MAYER-VIETORIS long exact sequence is the LES associated to the SES of chain complexes

$$\boxed{0 \to C_\bullet^I \longrightarrow C_\bullet^A \oplus C_\bullet^B \longrightarrow C_\bullet^K \longrightarrow 0}$$

where the first map includes each chain $\gamma \in C_\bullet^I$ to $C_\bullet A$ and $C_\bullet B$ separately as $(\gamma, \gamma)$; and the second map sends any pair $(\xi, \eta)$ to the difference $\xi - \eta \in C_\bullet^K$

b) So the M-V sequence is of the form

$$\cdots \xrightarrow{\Delta_{i-1}} H_i(I) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(K) \xrightarrow{\Delta_i} H_{i-1}(I) \longrightarrow \cdots$$

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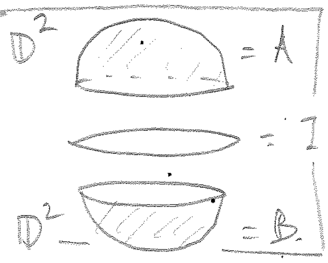
<u>So WHAT?</u> Well, we can compute the homology groups of ALL SPHERES AT ONCE, using only those of the circle! To see how, decompose

(n-disks)

$$\boxed{\begin{array}{l} \mathbb{S}^n = A \cup B, \quad \text{where } A \simeq \mathbb{D}^n \simeq B \\ \text{with } \quad A \cap B = \mathbb{S}^{n-1} \end{array}}$$




Now $H_i A \oplus H_i B = 0$ whenever $i > 1$, so we get

$$0 \longrightarrow H_i(\mathbb{S}^n) \xrightarrow{\Delta_i} H_{i-1}(\mathbb{S}^{n-1}) \longrightarrow 0$$

for $i \geq 1$ in the M-V long exact sequence. This forces $\Delta_i$ to be an isomorphism for all such $i$. Now induct on $n$ using the fact that $H_i(\mathbb{S}^1) = \left\{ \begin{array}{ll} \mathbb{F}, & i = 0 \text{ or } 1 \\ 0, & \text{else} \end{array} \right\}$ to get

$$H_i(\mathbb{S}^n) = \begin{cases} \mathbb{F}, & i = 0 \text{ or } n \\ 0, & \text{otherwise} \end{cases}$$

[The $i=0$ case is handled separately...]